# Exam - Statistics (WBMA009-05) 2020/2021

**Date and time**: November 2, 2020, 16.00-18.00h
**Place**: A. Jacobshal 02

**Rules to follow:**

- This is a closed book exam. Consultation of books and notes is **not** permitted. You can use a simple (non-programmable) calculator.
- Do not forget to write your name and student number onto each paper sheet. There are 2 exercises and you can reach 90 points. ALWAYS include the relevant equation(s) and/or short descriptions.
- **We wish you success with the completion of the exam!**

START OF EXAM

1. **Sample from the Exponential distribution.** [40]
Consider a random sample from the Exponential distribution with parameter $\theta > 0$,

$$X_1, \ldots, X_n \sim \text{EXP}(\theta)$$

The density and cumulative distribution function of the EXP($\theta$) distribution are

$$f_\theta(x) = \theta \cdot e^{-\theta x} \quad (x \geq 0) \quad \text{and} \quad F_\theta(x) = 1 - e^{-\theta x} \quad (x \geq 0)$$

(a) Provide a sufficient statistic for $\theta$. [5]

(b) Derive the Method of Moments (MM) estimator [2]
and show that the Maximum Likelihood (ML) estimator is $\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} X_i}$. [2]
Confirm via the 2nd derivative that $\hat{\theta}_{ML}$ is really a maximum point. [1]

(c) Show that the expected Fisher information (for $n = 1$) is $I(\theta) = \frac{1}{\theta^2}$. [5]

(d) Give the asymptotic distribution of the ML estimator. [5]

(e) Assume $n = 25$ and the mean realization: $\bar{X}_{25} = 4$. Use your result from (d) to derive an **asymptotic** 2-sided 90% confidence interval for $\theta$. [5]

(f) Assume $n = 25$ and the testproblem $H_0 : \theta = 0.5$ $\underline{vs.}$ $H_1 : \theta \neq 0.5$.
Based on your result from (d), for which realisations $\bar{X}_{25}$ would you reject $H_0$ to the level $\alpha = 0.05$? [5]

(g) Give an equation for the $q_\alpha$ quantile of the EXP($\theta$) distribution. [5]

(h) Assume $n = 1$ and the realization: $x_1 = 4$. Use your result from (g) to derive an **exact** 2-sided 90% confidence interval for $\theta$. [5]

**HINTS:**
(1) For $X \sim \text{EXP}(\theta)$ we have: $E[X] = \frac{1}{\theta}$ and $V(X) = \frac{1}{\theta^2}$.
(2) The Exponential distribution fulfills all regulatory conditions required for the asymptotic efficiency of the ML estimator.
(3) A table with quantiles of the standard Gaussian distribution is provided below.

2. **Sample from the Gaussian distribution** $\boxed{50}$
We consider a sample from a Gaussian distribution: $X_1, \dots, X_n \sim \mathcal{N}(\mu, \sigma^2)$ with density:

$$f_{\mu,\sigma^2}(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot \exp\left\{-\frac{1}{2} \cdot \frac{(x-\mu)^2}{\sigma^2}\right\} \qquad (x \in \mathbb{R})$$

We further assume that we know that $\mu = 0$, while $\sigma^2 > 0$ is an unknown parameter.

(a) Show that $\sum\limits_{i=1}^{n} X_i^2$ is a sufficient statistic for $\sigma^2$. $\boxed{5}$

(b) Compute the Maximum Likelihood (ML) estimator of $\sigma^2$. $\boxed{5}$
Check via the 2nd derivative whether you really have a maximum point. $\boxed{5}$

**Consider the simple test problem**: $H_0 : \sigma^2 = 1$ <u>**vs**</u> : $H_1 : \sigma^2 = 4$.

(c) Compute the corresponding density ratio $W(X_1, \dots, X_n) := \frac{f_{0,1}(X_1,\dots,X_n)}{f_{0,4}(X_1,\dots,X_n)}$. $\boxed{5}$

(d) Assume $n = 5$ and give the rejection region of the uniform most powerful (UMP) test to the level $\alpha = 0.05$.
Would the realization $\sum\limits_{i=1}^{5} X_i^2 = 10$ lead to a rejection of $H_0$? $\boxed{5}$

(e) Indicate how to compute the power of the UMP test at $\sigma^2 = 4$. Given the quantiles below, explain why the power is in between 0.5 and 0.9. $\boxed{5}$

(f) Assume $n = 5$ and the realization: $W(X_1, \dots, X_5) = 0.3$.
Check whether $H_0$ can be rejected to the test level $\alpha = 0.05$. $\boxed{5}$

**Now consider the more general test problem**: $H_0 : \sigma^2 = 1$ <u>**vs**</u> : $H_1 : \sigma^2 \neq 1$.

(g) Show that the corresponding likelihood ratio test (LRT) statistic is of the form $\lambda(X_1, \dots, X_n) = \{\hat{\sigma}^2_{ML}\}^a \cdot \exp\{-a \cdot \hat{\sigma}^2_{ML} + a\}$, where $a > 0$ is a constant. $\boxed{5}$

(h) For the sample size $n = 5$ give the two-sided rejection region of the LRT to the test level $\alpha = 0.1$. $\boxed{10}$

**HINTS:**
(1) For $X_1, \dots, X_n \sim \mathcal{N}(0,1) \;\Rightarrow\; X_1^2 + \dots + X_n^2 \sim \chi^2_n$.
(2) A table with quantiles of the $\chi^2_5$ distribution is provided below.
(3) For $a > 0$ the function $f(x) = x^a \cdot e^{-ax}$ is monotonically increasing for $0 < x < 1$ and monotonically decreasing for $x > 1$.

| $\alpha$ | 0.025 | 0.05 | 0.1 | 0.5 | 0.9 | 0.95 | 0.975 |
|---|---|---|---|---|---|---|---|
| $\mathcal{N}(0,1)$ | -1.96 | -1.64 | -1.28 | 0 | 1.28 | 1.64 | 1.96 |
| $\chi^2_5$ | 0.83 | 1.15 | 1.61 | 4.35 | 9.24 | 11.07 | 12.83 |

Table 1: Quantiles $q_\alpha$ of the $\mathcal{N}(0,1)$ and the $\chi^2_5$ distribution.